Constrained Optimization

$\min f(x)$

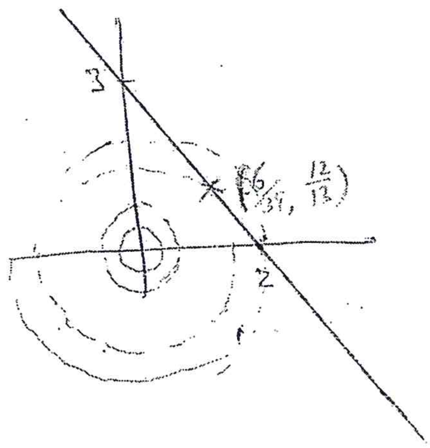
$\text{s.t. } g(x) = 0$

Supplementary Class on next Monday (December 1) from 2:35-4:15pm at LT6, Lady Shaw Building

Lagrange Multiplier method

$\min \mathcal{L}(x) = f(x) + \lambda g(x)$ unconstrained optimization

$$\begin{cases} \dfrac{\partial \mathcal{L}(x)}{\partial x} = f'(x) + \lambda g'(x) = 0 \\[2ex] \dfrac{\partial \mathcal{L}(x)}{\partial \lambda} = g(x) = 0 \end{cases}$$




__Eg 1.__ $\min x^2 + y^2$

$\text{s.t. } 3x + 2y = 6$

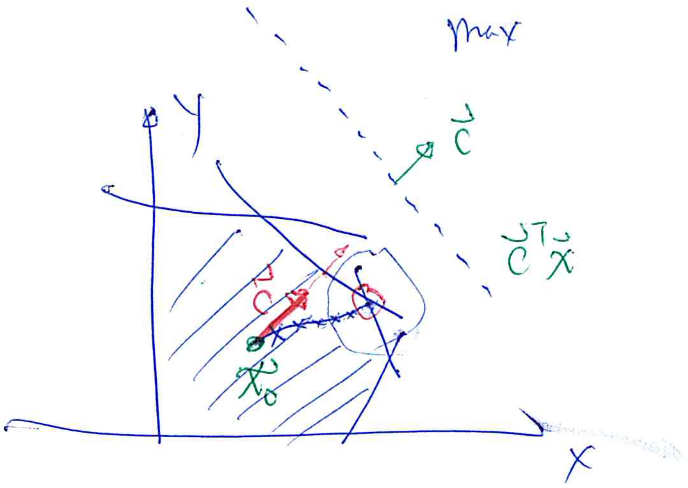
$\min \mathcal{L}(x, y) = x^2 + y^2 + \lambda(3x + 2y - 6)$

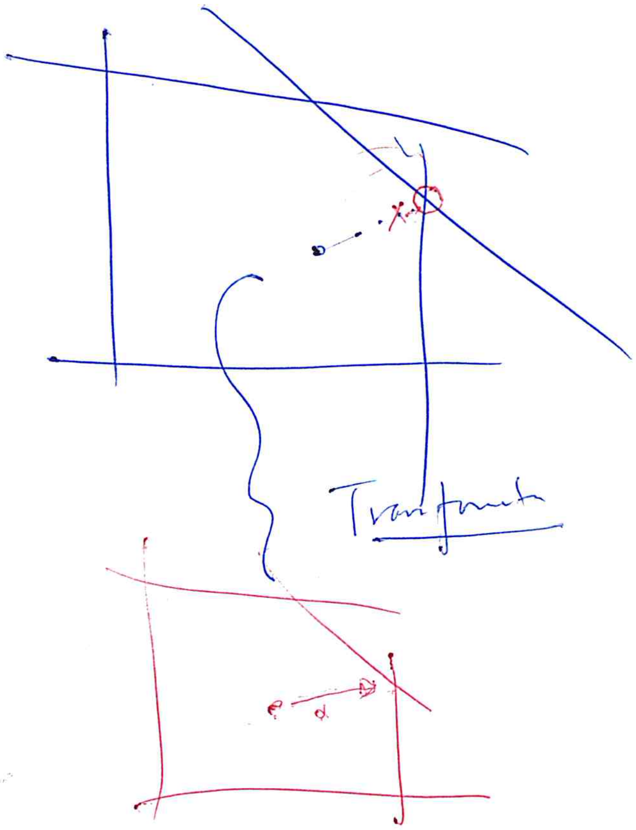
$\dfrac{\partial \mathcal{L}}{\partial x} = 2x + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}x \Rightarrow x = -\frac{3}{2}\lambda$

$\dfrac{\partial \mathcal{L}}{\partial y} = 2y + 2\lambda = 0 \Rightarrow \lambda = -y \Rightarrow y = -\lambda$

$\dfrac{\partial \mathcal{L}}{\partial \lambda} = 3x + 2y - 6 = 0 \Rightarrow 3(-\frac{3}{2}\lambda) + 2(-\lambda) = 6 \Rightarrow -\frac{13}{2}\lambda = 6 \Rightarrow \lambda = -\frac{12}{13}$

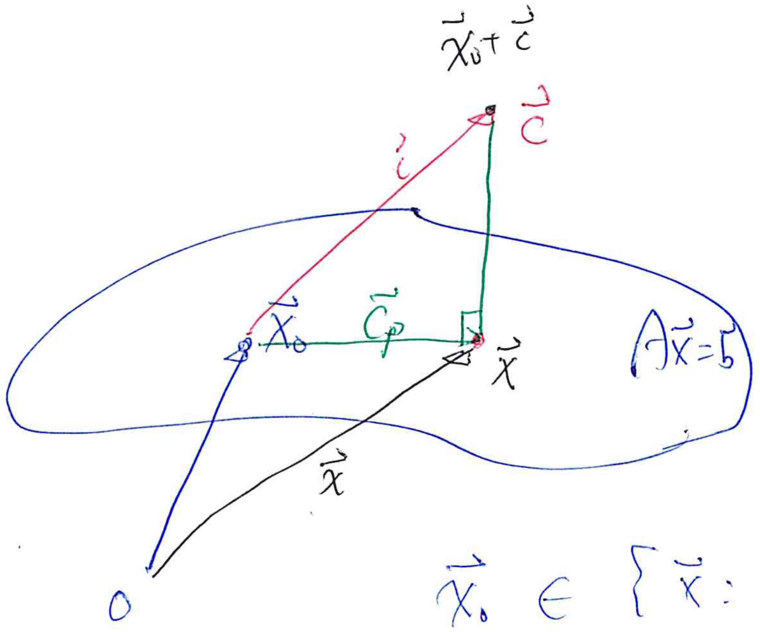
$\therefore x = +\frac{6}{39}, \quad y = +\frac{12}{13}$




rescale the problem

$$\vec{x}_i \longrightarrow \hat{x}_i = \left( \begin{array}{c} \vdots \end{array} \right) \boxed{\text{Centre pt}}$$




$\vec{x}_0 \in \{\vec{x} : A\vec{x}=\vec{b}\}$

$\Rightarrow A\vec{x}_0=\vec{b}$ ①

$$\min \frac{1}{2}\| \vec{x} - (\vec{x}_0 + \vec{c}) \|_2^2$$
$$A\vec{x} = \vec{b} = \vec{0}$$

Constrained optimization.

$\vec{x}_0 + \vec{c} \notin \{\vec{x} : A\vec{x}=\vec{b}\}$

$\vec{x}_0 + \alpha\vec{c}_p \in \{\vec{x} : A\vec{x}=\vec{b}\}$

$$g(\vec{x},\vec{\lambda}) = \frac{1}{2}\| \vec{x} - (\vec{x}_0+\vec{c})\|_2^2 + \vec{\lambda}^T(A\vec{x}-\vec{b})$$ unconstrained

$$\begin{cases} \dfrac{\partial g}{\partial \vec{x}} = \vec{x} - (\vec{x}_0+\vec{c}) + (\vec{\lambda}^T A)^T = 0 & ① \\ \dfrac{\partial g}{\partial \lambda} = A\vec{x} - \vec{b} = 0 & ② \end{cases}$$

① $\Longrightarrow$ $x = \vec{x}_0 + \vec{c} - A^T\vec{\lambda}$ ③

$\Rightarrow$ ③ $\rightarrow$ ② $\quad \underbrace{A(\vec{x}_0 + \vec{c})}_{\vec{b}} - AA^T\vec{\lambda} = \cancel{\vec{b}}$

$$\Rightarrow \vec{\lambda} = (AA^T)^{-1}A\vec{c} \quad ④$$

④ $\rightarrow$ ③ $\quad \vec{x} = \vec{x}_0 + \vec{c} - A^T(AA^T)^{-1}A\vec{c}$

$$= \vec{x}_0 + \underbrace{(I - A^T(AA^T)^{-1}A)\vec{c}}_{\vec{c}_p}$$ #

$$\max \quad z = x_1 + 2x_2 + 0x_3$$

$$x_1 + x_2 + x_3 = 8$$

$$x_1, x_2, x_3 \ge 0$$

$\vec{x}_0 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \in FR_{\text{original system}}$ $\quad \hat{x}_1 = \frac{x_1}{2} \quad \hat{x}_2 = \frac{x_2}{2} \quad \hat{x}_3 = \frac{x_3}{4}$

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in FR_{\text{transformed system}}$$

$$\boxed{\max \begin{cases} z = 2\hat{x}_1 + 4\hat{x}_2 + 0\hat{x}_3 \\ 2\hat{x}_1 + 2\hat{x}_2 + 4\hat{x}_3 = 8 \\ \hat{x}_1, \hat{x}_2, \hat{x}_3 \ge 0 \end{cases}}$$

$\hat{A} = (2\ 2\ 4)$

$\hat{c} = (2, 4, 0)$

$\hat{c}_p = (I - \hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{A})\hat{c}$

$$\vec{\tilde{x}}_1 = \vec{\tilde{x}}_0 + \alpha \vec{\tilde{c}}_p$$

$$= \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}}_{\text{always satisfies } \hat{A}\hat{x} = \hat{b}}$$

$\hat{A}(\hat{c}_p) = \vec{0} \Leftrightarrow$ always satisfies

at most $\boxed{\alpha = \frac{1}{2}} \Rightarrow \vec{x}_1 \ge \vec{0}$

$\boxed{\text{Karmarkar's choose } \frac{1}{2}\alpha_{max}, \frac{1}{4}\alpha_{max}}$